Modeling Malthusian Dynamics in Pre-industrial Societies

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ABSTRACT

The discussion about the Malthusian character of pre-industrial economies that has arisen in the recent years extensively uses simple mathematical models. This article analyzes some of these models to determine their conformity with Malthusian postulates. The author suggests two models that are more adequate for the description of the Malthusian patterns.

Until recently, most economic historians were inclined to think that the medieval economies in Eurasia had a Malthusian nature (Allen 2008: 951). However, following the publication of Lee and Anderson's work (Lee and Anderson 2002), many came to dispute this opinion. A discussion has arisen about how the available data confirm the Malthusian relationship between demographic dynamics and consumption (*i.e.*, real wages). This discussion has largely involved simple mathematical models of the Malthusian economy.

In 1980, Lee published the first and most popular of these models. This model describes the relationship between the real wage, w_t (consumption), and labor resources, N_t (population), using the following equation:

$$w_t = \exp(\mu + \rho t + \varepsilon_t)\, N_t^{-\eta}. \qquad (1)$$

Or, in the logarithmic form:

$$\ln w_t = \mu + \rho t - \eta \ln N_t + \varepsilon_t. \qquad (2)$$

Here, t is time; $\mu,\ \rho,\ \eta$ are some non-negative constants; and ε_t is a variable that takes into account the climatic effect and other exogenous parameters. The factor ρ describes the capital increase and technological advances, thus the Malthusian economy features $\rho = 0$.

If we take into account that in the ideal case $\varepsilon_t = 0$, the model can be expressed with quite a simple equation: $w_t = C N_t^{-\eta}$, where C is a certain constant. The drawbacks of this equation are evident: a small population N_t results in a consumption rate close to infinity, while in a large population the consumption becomes too small to ensure subsistence. Besides, this equation only shows the relationship between population and consumption. The model contains no feedback to demonstrate in what way the consumption influences population growth.

Wood (1998) suggests a feedback option for which he derives an equation basing on the same equation (1) as Lee, but formulates it as follows:

$$w_t = \theta \, (S_t/N_t)^{\eta} \, . \tag{1a}$$

Here, θ is the minimum per capita consumption rate, and S_t is the maximum population that can subsist in the given territory when the consumption equals θ. S_t can grow with technological advances, but the Malthusian case features a constant S_t, $S_t = S_0$. Wood believes that the birth rate b_t and death rate d_t can be described with the following equations:

$$b_t = \beta_0 + \beta_1 \ln w_t + \beta_2 \, d_t \tag{3}$$
$$d_t = \delta_0 + \delta_1 \ln w_t + \delta_2 \, b_t \tag{4}$$

where β_0, β_1, β_2, δ_0, δ_1, and δ_2 are certain constants. Thus, the equation below describes population growth:

$$dN_t/dt = (b_t - d_t)N_t \tag{5}$$

Deriving b_t and d_t from the system of Equations (3)–(4) and inserting them into (5) yields:

$$dN_t/dt = (b_t - d_t)N_t = (c_0 + c_1 \ln w_t)N_t$$

where c_0 and c_1 are certain constants. Substituting equation (1a) here produces:

$$dN_t/dt = (c_2 + c_3 \ln N_t)N_t \tag{6}$$

where c_2 and c_3 are certain constants. The differential equation (6) has the time-independent solution $N_t = N_0 = \exp(-c_2/c_3)$; its chart will be a horizontal line. According to the theorem of the unique existence of the solution, no other solutions (integral curves) may cross this horizontal line. The derivative dN_t/dt is positive below this line, in the area $0 < N_t < N_0$; the solutions monotonically increase and the integral curves approximate the horizontal line. The derivative dN_t/dt is negative above this line; the solutions monotonically decrease and the integral curves approximate

the horizontal line from above. Finally, the solutions cannot oscillate: the population cannot first feature growth and then loss due to 'the Malthusian crises'. Wood justifies this behavior of his model stating that the Malthusian crises 'are not a necessary feature of Malthusian systems… This conclusion is contrary to the belief of many economic historians (*e.g.*, Le Roy Ladurie 1974: passim; Postan and Hatcher 1985: 69) though not to anything that Malthus himself ever wrote' (Wood 1998: 110).

However, Malthus did write about population loss and depopulation:

> The power of population is so superior to the power of the earth to produce subsistence for man that premature death must in some shape or other visit the human race. The vices of mankind are active and able ministers of depopulation. They are the precursors in the great army of destruction, and often finish the dreadful work themselves. But should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague advance in terrific array, and sweep off their thousands and tens of thousands. Should success be still incomplete, gigantic inevitable famine stalks in the rear, and with one mighty blow levels the population with the food of the world (Malthus 1798: 61).

Wood's model does not, therefore, describe the population dynamics envisioned by Malthus himself. Nevertheless, it is used in many studies devoted to the analysis of the Malthusian economy in traditional societies.

Sometimes an iterative version of this model is used, implying calculations on an annual basis. Equation (1a) in the version put forth by Møller and Sharp (2009) has logarithmic form:

$$\ln w_t = c_0 - c_1 \ln N_t + \ln A \qquad (2a)$$

Birth and death rates are calculated from the simplified equations:

$$b_t = a_0 + a_1 \ln w_t \qquad (3a)$$
$$d_t = a_2 - a_3 \ln w_t \qquad (4a)$$

The population N_t is related to the population N_{t-1} in the previous year through the following relationship:

$$\ln N_t = \ln N_{t-1} + b_{t-1} - d_{t-1} \qquad (7)$$

here, A, c_0, c_1, a_0 … a_3 are certain constants. Inserting (3a) and (4a) into (7) gives:

$$\ln N_t = \ln N_{t-1} + (a_1 + a_3)\ln W_{t-1} + a_0 - a_2 =$$
$$= \ln N_{t-1} + (a_1 + a_3)(c_0 - c_1 \ln N_{t-1} + \ln A) + a_0 - a_2 = u \ln N_{t-1} + \ln v$$

where u and v are certain constants. The resulting equation is:

$$N_t = v(N_{t-1})^u \tag{8}$$

This equation generates a series of population values. If the population at the initial moment equals a million (*i.e.*, $N_1 = 1$), then $N_2 = v$, N_3 equals v raised to a power of $1 + u$; N_3 equals to v raised to a power of $1 + u + u^2$, *etc*. If $|u| > 1$, then $N_t \to \infty$, which is impossible under the condition of limited resources in the Malthusian theory. If $0 < u < 1$, then N_t monotonically tends to a finite bound. Finally, the case $-1 < u < 0$ produces a very specific series in which the population increases in even years and decreases in odd years (or vice versa). Thus, the Møller-Sharp model has the same drawback as Wood's initial model: it cannot describe long-term population oscillations.

Another iterative version of the model is the one developed by Ashraf and Galor (2011). Beginning with equation (1a), the authors of this model take into consideration the number of adults and children, and optimize expenses. They, nevertheless, ultimately come to the same equation (8).

One more version of Wood's model is that of Voigtlander and Voth (2009). They use equation (1a), but replace equations (3) and (4) with (3a) and (4a):

$$b_t = b_0 (w_t / \theta)^m \tag{3a}$$
$$d_t = d_0 (w_t / \theta)^n \tag{4a}$$

where b_0 and d_0 are certain constants. Inserting (1a) into equation (5) yields:

$$dN_t/dt = (b_t - d_t)N_t = (b_0(S_0/N_t)^{\eta m} - d_0(S_0/N_t)^{\eta n})N_t =$$
$$= q(p - N_t^{\eta(m-n)})N_t^{1-\eta m} \tag{6a}$$

where p and q are certain constants. The differential equation (6a) has the time-independent solution $N_t = N_0 = p^{1/\eta(m-n)}$, which represents a horizontal line. As with the above model, the solution curves that are beneath this line monotonically increase and those lying above the line – monotonically decrease. Thus, this model has the same limitation as Wood's model and its other derivatives: it does not offer oscillating solutions.

Brander and Taylor (1998) have suggested another popular model. This model analyzes some abstract renewable resource con-

sumed in the course of human activities. For example, it might be forest resources or soil yield. S_t is the available amount of this resource (in year t), and K denotes its reserve in nature. The equation for consumption of this resource is as follows:

$$dS_t/dt = rS_t(1-S_t/K) - uS_tN_t \qquad (8)$$

where r and u are certain constants. The first term on the right side describes the process of natural resource renewal; the second term describes resource depletion owing to economic activity. The population number is given by the following equation:

$$dN_t/dt = (d + vS_t)N_t \qquad (9)$$

where d and v are constants, and $d < 0$ in this case. This equation shows that natural population growth depends on the availability of resource S_t.

Brander and Taylor have shown that the system of equations (8)–(9) has oscillating solutions: when the resource is abundant the population grows, when it is exhausted the population decreases until the resource is renewed. Brander and Taylor refer to their model as 'Malthusian-Ricardian'. Initially, the model was intended to describe the economy of Easter Island, but afterwards it got wider application as a sufficiently general model of the Malthusian economy (*e.g.*, Maxwell and Reuveny 2000; D'Alessandro 2007). It is essential to note, however, that resource S_t in the Brander–Taylor model is not the harvest gathered by farmers. According to Brander and Taylor, the crop is denoted by the term uS_tN_t and it is *deducted* from the resource S_t. According to Szulga (2012), such a model describes a society of gatherers (or hunters) rather than an agrarian society. Meanwhile, Malthus mainly studied agricultural economies. Thus, Brander-Taylor model cannot be referred to as a 'Malthusian-Ricardian' one.

Up to this point, I have confined my discussion to the analysis of simple models of the Malthusian economy that consist of no more than two differential equations. Naturally, there exist more complicated models (*e.g.*, Usher 1989; Komlos and Artzrouni 1990; Chu and Lee 1994; Galor and Weil 2000; Lee and Tuljapurkar 2008) that allow for better behavioral freedom and offer oscillating solutions, as well. Many such models have been constructed within the framework of cliodynamic research actively performed in Russia and the USA (*e.g.*, Tsirel 2004; Korotayev, Malkov, and Khaltourina 2005, 2006; Korotayev, Malkov, and Grinin 2007;

Turchin 2007, 2009; Malkov 2009). However, almost all models described in literature have the same drawback: they contain uncertain coefficients whose values are unknown and cannot be determined in principle. The more complicated is a model, the more uncertain coefficients it contains. Meanwhile, these coefficients determine the behavior of the model, and different coefficient values result in different population dynamics. Thus, an uncertainty originates: as coefficient values are unknown, it is also uncertain which of the possible behavioral variants corresponds to historical reality and which of them could not possibly be realized.

In the remainder of this article, I would like to discuss two simple models that contain no uncertain parameters and, in my opinion, are sufficiently adequate for the description of the Malthusian population dynamics. Within the first model, N_t is the population in the year t, as above; K_t is the grain stock after the harvest estimated in terms of minimum annual per caput grain consumption rates (rate of consumption is assumed about 240 kg); and r is the natality under favorable conditions. The area under cultivation and the harvest depend on the population, and with the population growth they tend to the constant determined by the maximum area under cultivation maintained by the agricultural community. We will consider that the harvest is determined by the equation $P_t = aN_t /(N_t+d)$, where a and d are certain constants. To describe the population dynamics we use the standard logistic equation:

$$dN_t/dt = rN_t (1 - N_t / K_t) \qquad (10)$$

K_t in this logistic equation denotes the carrying capacity (*i.e.*, the maximum size of population that may live in this territory). In our case, this population size corresponds to the number of minimum annual consumption rates in storage. Annually, N_t consumption rates are consumed, and the stock growth will be equal to:

$$dK_t/dt = P_t - N_t = aN_t/(N_t+d) - N_t \qquad (11)$$

Thus, we have the simplest system of two differential equations (10)–(11). This system has an equilibrium state, when the population and stock remain constant, namely, when $K_0 = = N_0 = a - d$.

If N in the equation for dP/dN tends to 0, we will obtain the harvest a/d (number of consumption rates) gathered by one farmer in favorable conditions (when the population is small and he or she is able to cultivate the maximum area). Thus, the value $q = a/d$ shows how many households one farming family can support. The history

of agricultural societies shows that q usually varies within the limits *1.2 < q < 2*. We can express a and d in terms of q and N_0:

$$d = N_0 / (q - 1), \quad a = qN_0 / (q - 1)$$

N_0 can be conventionally set equal to 1 and thus, there are two constants in this model, r and q, which have physical significance and vary within the known limits: *0.01 < r < 0.02,* and *1.2 < q < 2.* The common methods used for investigation of dynamic systems allow us to determine that system (10)–(11) describes dying oscillations. The first oscillations can have different periods; however, when the curve approaches the equilibrium state, the period is close to:

$$T = 2\pi / \sqrt{(r - r/q - r^2/4)}$$

The period T decreases when r and q increase, and increases, respectively, when these values decrease (Table 1 and Fig. 1).

Table 1

Period of oscillations with various r and q (in years)

$q \backslash r$	0.01	0.02
1.2	154	110
2.0	89	63

Thus, the period of oscillations in this model is comparable to the duration of secular demographic cycles observed in the history of many states (Turchin and Nefedov 2009).
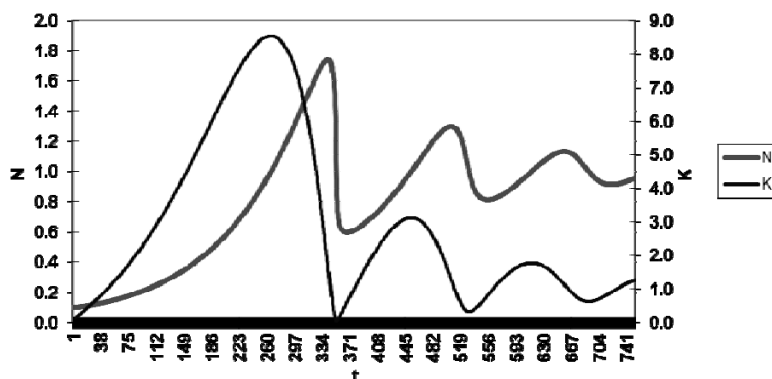

Fig. 1. Example of calculations using the model (r = 0.01; p = 1.2)

The dynamics of the agricultural population according to this model have an oscillating nature. In theory these oscillations die

out and the system tends to the equilibrium state, but various random impacts and influences neglected herein (*e.g.*, catastrophic crop failure) disturb the system equilibrium, after which there starts a new series of dying oscillations. The peculiar feature of the agricultural society is that its economic dynamics substantially depend on such a random value as the crop yield. The random factors that impact such systems are generally assumed to be exogenous; however, the dependence on crop yield variations is an intrinsic, endogenous feature of agricultural production. Therefore, one arrives at the conclusion that a special random value describing crop yield must be incorporated into the ideal model of the Malthusian cycle. This can be conveniently done within the iterative model where the calculations are made from year to year.

For convenience, I consider production years to start with the harvest, not a specific calendar date. The population number N_t at the beginning of year t is expressed in terms of the number of households or families (conventionally assuming that a household contains five people). In theory (*i.e.*, when there is enough land for cultivation), a farming household cultivates a standard parcel of land (*e.g.*, a Middle Eastern 'çiftlik') and one can measure the maximum possible area of arable land in terms of standard parcels S. When the number of households N_t exceeds S, two families can live on some parcels.

Let a_t represent the annual crop yield t, expressed in terms of minimum family grain consumption rates that can be gathered on a standard parcel. We will express the crop yield in the form $a_t = a_0 + d_t$, where a_0 is the average crop yield, d_t is a random value that accepts values from the segment $(-a_1, a_1)$. The value a_1 is less than a_0 and the crop yield a_t varies within the interval of $a_0 - a_1$ to $a_0 + a_1$. With the units of measure that I have assumed, the harvest Y_t (number of consumption rates) can be expressed in the following simple form:

$$Y_t = a_t N_t \text{ if } N_t < S, \text{ and } Y_t = a_t S \text{ if } N_t > S.$$

If there is a grain surplus in the year t, that is per-capita production $y_t = Y_t/N_t$ exceeds the 'satisfactory consumption' p^1 ($p^1 > 1$), then the farmers do not consume all produced grain, but store up the surplus (for the sake of simplicity we will assume that they store half of the surplus). However, it is worth noting that, owing to the storage conditions, the household stock Z_t cannot grow to infinity and is

limited by certain value Z^0. If there are surpluses exceeding this value, they all are consumed. If the year is lean and the production y_t falls below the level p^1, the farmers take grain from the stock, increasing the consumption, if possible, up to the level p^1. If the stock is not sufficient, it is consumed to the full.

The population growth rate r_t is determined as the ratio of the population N_{t+1} in the following year to the population N_t in the previous year. The growth rate r_t depends on the consumption p_t. When the consumption is equal to the minimum normal rate ($p_t = 1$), the population remains constant ($r_t = 1$). I designate the maximum natural growth r^0, and the consumption rate needed to ensure it as p^0. I believe that $r^0 = 1.02$, that the maximum annual population growth is two per cent. When $1 < p_t < p^0$, population growth is linearly dependent on consumption, and in the case when $p_t > p^0$, it does not increase ($r = r^0$). For $p_t < 1$, the dependence of r_t on p_t is taken as $r_t = p_t$ (*i.e.*, in case of crop failure the surviving population will be equal to the number of consumption rates and all people that do not have a sufficient annual food reserve will perish from starvation). Consequently, the population in the following year will be $N_{t+1} = r_t N_t$.

Considering the typical case from the Middle East or Russia from the sixteenth to the eighteenth centuries, when every family could obtain two minimal consumption rations from one standard parcel, one can assume $a_0 = 2$ for the numerical experiment. The scatter of crop yield (ratio a_1/a_0) was large enough (*e.g.*, it was about 60 per cent of average crop yield in Egypt). Hence, it appears that one can assume $a_1 = 1.2$. As for the random value d_t, it may be approximated using squared uniform distribution: if w is a value uniformly distributed over the segment (−1.1), then this random value can be taken as $d_t = a_1 w^2$ sign(w) (Nefedov and Turchin 2007). The maximum number of standard parcels S can be conventionally assumed to equal one million, and the maximum stock to equal ten-year ones *(Z^0 = 10)*. Here, I consider a case in which farmers call upon the experience gained by preceding generations and start laying the crop up in storage as soon as the per-capita production exceeds 1.05 of the minimum level ($p^1 = 1.05$). This calculation has an idealized character, allowing one to assume $N_1 = 0.8$ as the initial population value (in year $t = 1$). As the calculation results depend on a random value (*i.e.*, crop yield), they can vary with each pro-

gram run. Despite this variation, one can qualitatively observe a pattern of demographic cycles that seems typical: population growth periods alternate with demographic catastrophes. The duration of this cycle is, as in the previous model, 80–150 years (Fig. 2).
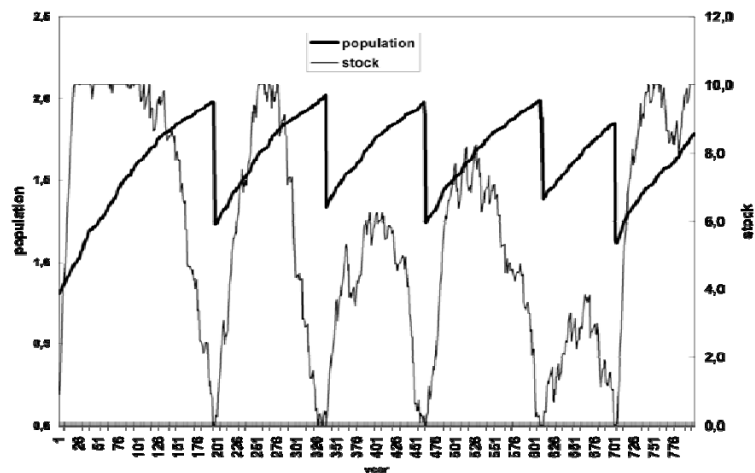

Fig. 2. Example of calculation using this model for r^0 =1.02, p^0 = 2, a_0 = 2, a_1=1.2, p_1=1.05

Naturally, this model describes just the basic mechanism of the demographic cycle omitting many details (*e.g.*, the existence of the state and military elite, the emergence of large landowners). Such factors are taken into account in other models (*e.g.*, Nefedov and Turchin 2007) and the calculations made using these models show that the qualitative cyclic pattern changes insignificantly compared to the suggested model. On the whole, it seems quite certain that the availability of grain stock in farms contributes to a long-term economic stabilization. However, the population growth results in the stock depletion, and, sooner or later, major harvest failures provoke catastrophic starvations followed by events like epidemics, uprisings of starving people, and/or invasions by external enemies seeking to take advantage. As a result, the population size can decrease even by half and a new demographic cycle starts. While the model calculations suggest that this new cycle might start immediately after the catastrophe, in real life such crises as wars and uprisings have some inertia and impede economic revival. In such cases, stabilization is delayed.

Finally, it is worth noting that after the publication of Wood's model economic historians came to consider the Malthusian economy as a system where the population size cannot exceed the carrying capacity and, consequently, the 'Malthusian crisis' is impossible. For example, Read and LeBlanc (2003: 59) suggest that

> … there is a standard model for the pattern of human population growth and its relationship to carrying capacity (K), namely, that most of the time human populations have low to nonexistent rates of growth… The model is often implicit and may simply assert that, until recently, population sizes have always been well below K and growth rates very low.

But Le Roy Ladurie, Postan, Hatcher, and many other economic historians insist that 'the Malthusian crises' were quite common phenomena in lived history, a fact acknowledged by Wood himself. The models described in this article show that the inevitability of similar crises arises from the simple laws that rule the functioning of agrarian economies.

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